

AN EXTENSION RESULT OF CR FUNCTIONS BY A GENERAL SCHWARZ REFLECTION PRINCIPLE

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ABSTRACT. It is known that a real analytic CR function f on a real analytic, generic submanifold M in \mathbb{C}^N can be holomorphically extended. A stronger result on a finite type, real analytic, generic submanifold M is found in which we assume f a continuous CR function with real analytic imaginary part $Im(f)$. The idea is contained in a general Schwarz Reflection Principle in one complex variable.

1. A general Schwarz Reflection Principle

The classical Schwarz Reflection Principle can be stated as follows:

Theorem 1.1. Suppose that Ω is a connected domain, symmetric with respect to the real axis, and that $L = \Omega \cap \mathbb{R}$ is an interval. Let $\Omega^+ = \{z \in \Omega : Im(z) > 0\}$. Suppose that $f \in A(\Omega^+)$, a function holomorphic on Ω^+ and that $Im(f)$ has a continuous extension to $\Omega^+ \cup L$ that vanishes on L . Then there is a $F \in A(\Omega)$ such that $F = f$ in Ω^+ and $F(z) = \overline{f(\bar{z})}$ in $\Omega - \Omega^+$.

Our generalized Schwarz Reflection Principle (Theorem 1.2) replaces the vanishing of $Im(f)$ on L by the real analyticity of $Im(f)$ on L .

Theorem 1.2. Let $D = D(0, 1)$ be the open disc centered at the origin with radius 1, $D^+ = \{z \in D : Im(z) > 0\}$, $L = D \cap \mathbb{R}$. Suppose that $f \in A(D^+)$ and $Im(f) \in C(D^+ \cup L)$ such that $v(x, 0) = Im(f)|_L$ is real analytic at 0 with radius of convergence r . Denote $D_r = D(0, r)$, then there exists $F \in A(D^+ \cup D_r)$ such that

$$(1.1) \quad F(z) := \begin{cases} f(z) & \text{for } z \in D^+ \\ \frac{f(z)}{f(\bar{z})} + 2iv(z, 0) & \text{for } z \in D_r - D_r^+ \end{cases}$$

where $v(z, 0)$ denotes a holomorphic function on D_r .

Proof: Since $v(x, 0) = Im(f)|_L$ is real analytic at 0 with radius of convergence r , we can write $v(x, 0) = \sum a_n x^n$, $\forall |x| < r$. Thus we get a holomorphic function $v(z, 0) = \sum a_n z^n$ on D_r by the complexification of the variable x . Now $Im(f(z) - iv(z, 0))|_{(-r, r)} \equiv 0$, by the Schwarz Reflection Principle, the function

$$(1.2) \quad g(z) := \begin{cases} \frac{f(z) - iv(z, 0)}{f(\bar{z}) - iv(\bar{z}, 0)} & \text{for } z \in D_r^+ \\ \frac{f(z) - iv(z, 0)}{f(\bar{z}) - iv(\bar{z}, 0)} & \text{for } z \in D_r - D_r^+ \end{cases}$$

is holomorphic on D_r . Thus the function

$$(1.3) \quad F(z) := g(z) + iv(z, 0) = \begin{cases} \frac{f(z)}{f(\bar{z})} & \text{for } z \in D^+ \\ \frac{f(z)}{f(\bar{z}) - iv(\bar{z}, 0)} + iv(z, 0) & \text{for } z \in D_r - D_r^+ \end{cases}$$

is holomorphic on $D^+ \cup D_r$ and is the desired extension. Since a_n are real, $F(z) = \overline{f(\bar{z})} + 2iv(z, 0)$, $\forall z \in D_r - D_r^+$ which gives (1.1). \blacksquare

Theorem 1.2 leads to a reflection principle of harmonic functions.

Corollary 1.3. Let D , D^+ , L as above. Suppose that $v(x, y) \in C(D^+ \cup L)$ is harmonic in D^+ such that $v(x, 0)$ is real analytic at 0 with radius of convergence r . Denote $D_r = D(0, r)$, then there exists V harmonic in $D^+ \cup D_r$ such that

$$(1.4) \quad V(x, y) := \begin{cases} v(x, y) & \text{for } z \in D^+ \\ 2\operatorname{Re}(v(z, 0)) - v(x, -y) & \text{for } z \in D_r - D_r^+ \end{cases}$$

where $v(z, 0)$ denotes a holomorphic function on D_r .

Proof: Since we can find $u(x, y)$ in D^+ such that $u + iv \in A(D^+)$, we then apply Theorem 1.2 to get the extension $V(x, y)$ in D_r , $V(x, y) = \operatorname{Im}(\overline{f(\bar{z})} + 2iv(z, 0)) = 2\operatorname{Re}(v(z, 0)) - v(x, -y)$. \blacksquare

Remark 1.4. Since every real analytic curve in \mathbb{C} is locally biholomorphic to the real line. The general Schwarz Reflection Principle holds in the following case:

Let S be a real analytic curve in \mathbb{C} through 0, U a small connected neighborhood of 0 such that $U - S$ consists of two components U^+ , U^- . If $f \in A(U^+)$, the imaginary part $\operatorname{Im}(f) \in C(U - U^-)$ such that $\operatorname{Im}(f)|_{U \cap S}$ is real analytic, then f can be extended to a function F holomorphic in a neighborhood of 0 in \mathbb{C} .

2. Holomorphic extension of CR functions on a real analytic, generic CR submanifold in \mathbb{C}^N

We first introduce the necessary notations and definitions needed in the sequel. We mainly follow [1]. For $Z \in \mathbb{C}^N$, we write $Z = (z_1, \dots, z_n)$ where $z_j = x_j + iy_j$; we write $\bar{Z} = (\bar{z}_1, \dots, \bar{z}_n)$, where $\bar{z}_j = x_j - iy_j$, the complex conjugate of z_j . We identify \mathbb{C}^N with \mathbb{R}^{2N} , and denote a function f on a subset of \mathbb{C}^N as $f(x, y)$, or, by abuse of notation, as $f(Z, \bar{Z})$.

A smooth (real analytic) real submanifold of \mathbb{C}^N of codimension d is a subset M of \mathbb{C}^N such that $\forall p_0 \in M$, there is a neighborhood U of p_0 and a smooth (real analytic) real vector-valued function $\rho = (\rho_1, \dots, \rho_d)$ defined in U such that

$$(2.1) \quad M \cap U = \{Z \in U : \rho(Z, \bar{Z}) = 0\},$$

with differentials $d\rho_1, \dots, d\rho_d$ linearly independent in U .

Definition 2.1. A real submanifold $M \subset \mathbb{C}^N$ is CR if the complex rank of the complex differentials $\partial\rho_1, \dots, \partial\rho_d$ is constant for $p \in M$. It is generic if $\partial\rho_1, \dots, \partial\rho_d$ are \mathbb{C} linearly independent for $p \in M$.

Let $M \subset \mathbb{C}^N$ be a germ of real analytic, generic, CR submanifold of codimension d at p_0 , write $N = n + d$, $n \geq 1$. After a local holomorphic change of coordinates,

there exists Ω , a sufficiently small open neighborhood of 0 in \mathbb{C}^{n+d} , such that M is given in Ω by

$$(2.2) \quad \text{Im}(w) = \phi(z, \bar{z}, \text{Re}(w)),$$

with $z \in \mathbb{C}^n$, $w \in \mathbb{C}^d$, ϕ a real-valued analytic function (power series) of $z, \bar{z}, \text{Re}(w)$ such that $\phi(0, 0, 0) = 0$ and $d\phi(0, 0, 0) = 0$. Such a choice of coordinates is called regular coordinates.

Now suppose M is a real analytic, generic submanifold, locally parametrized by the regular coordinates near $0 \in M$. We extend the parametrization $\Psi(z, \bar{z}, s) = (z, s + i\phi(z, \bar{z}, s)) \in \mathbb{C}^{n+d}$ to a local real analytic diffeomorphism

$$(2.3) \quad \tilde{\Psi}(z, \bar{z}, s + it) = (z, s + it + i\phi(z, \bar{z}, s + it)) \in \mathbb{C}^{n+d},$$

defined in a neighborhood of $0 \in \mathbb{R}^{2n+2d}$.

By the above notation, we have the following holomorphic extension results, see [1].

Proposition 2.2. Let M be a real analytic generic submanifold of \mathbb{C}^{n+d} of codimension d in regular coordinates near $0 \in M$. If h be a CR function on M , then h extends holomorphically to a full neighborhood of 0 if and only if there exists $\epsilon > 0$ such that for every $z \in \mathbb{C}^n$, $|z| < \epsilon$, the function $s \mapsto h \circ \Psi(z, \bar{z}, s)$ extends holomorphically to the open set $\{s + it \in \mathbb{C}^d, |s| < \epsilon, |t| < \epsilon\}$ in such a way that the extension $H := h \circ \Psi(z, \bar{z}, s + it)$ is a bounded, measurable function of all its variables.

Corollary 2.3. Let $M \subset \mathbb{C}^N$ be a real analytic generic submanifold and f a CR function in a neighborhood of $p \in M$. Then f extends as a holomorphic function in a neighborhood of p in \mathbb{C}^N if and only if f is real analytic in a neighborhood of p in M .

According to Corollary 2.3, a CR function $f = u + iv$ on a real analytic, generic submanifold can be holomorphically extended to a full neighborhood if and only if u, v are real analytic on M . Our main theorem, to some extent, reduces the case to u continuous and v real analytic on M . We state it as follows:

(Main theorem) Let $M \subset \mathbb{C}^{n+d}$ be a real analytic, generic submanifold. $0 \in M$. In a neighborhood Ω of 0 in \mathbb{C}^{n+d} , M is given in regular coordinates (2.2), $\rho := \text{Im}(w) - \phi(z, \bar{z}, \text{Re}(w)) = 0$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$. Let $f = u + iv$ be a holomorphic function defined in the wedge $W = W(\Omega, \rho, \Gamma)$, with Γ an open convex cone in \mathbb{R}^d . If f extends continuously up to the edge $M \cap \Omega$ with $v = \text{Im}(f)$ is real analytic on the edge $M \cap \Omega$. Then there exists a holomorphic function F in a neighborhood $\tilde{\Omega}$ of 0 in \mathbb{C}^{n+d} such that $F|_{W \cap \tilde{\Omega}} = f$.

Let M be a generic submanifold of \mathbb{C}^{n+d} of codimension d and $p_0 \in M$. Let $\rho = (\rho_1, \dots, \rho_d)$ be a defining functions of M near p_0 and Ω a small neighborhood of p_0 in \mathbb{C}^{n+d} in which ρ is defined. If Γ is an open convex cone with vertex at the

origin in \mathbb{R}^d , we define

$$(2.4) \quad W(\Omega, \rho, \Gamma) := \{Z \in \Omega : \rho(Z, \overline{Z}) \in \Gamma\}.$$

The above set is an open subset of \mathbb{C}^{n+d} whose boundary contains $M \cap \Omega$. Such a set is called a wedge of edge M in the direction Γ centered at p_0 .

The following shows that $W(\Omega, \rho, \Gamma)$ is in a sense independent of the choice of ρ , which will allow us to change defining functions freely. See [1].

Proposition 2.4. Let ρ and ρ' be two defining functions for M near p_0 , where M and p_0 are as above. Then there is a $d \times d$ real invertible matrix B such that for every Ω and Γ as above the following holds. For any open convex cone $\Gamma_1 \subset \mathbb{R}^d$ with $\{Bx : x \in \Gamma_1\} \cap S^{d-1}$ relatively compact in $\Gamma \cap S^{d-1}$ (where S^{d-1} denotes the unit sphere in \mathbb{R}^d), there exists Ω_1 , an open neighborhood of p_0 in \mathbb{C}^{n+d} , such that

$$(2.5) \quad W(\Omega_1, \rho', \Gamma_1) \subset W(\Omega, \rho, \Gamma).$$

Below is an Edge-of-the-Wedge Theorem, see [2], [3]. Since the construction of the holomorphic extension in this theorem is essential to the proof of the main theorem, so we supply a proof of the Edge-of-the-Wedge Theorem from [2], p.157-159.

Theorem 2.5. (Edge-of-the-Wedge Theorem) If $\Gamma \subset \mathbb{R}^d$ is an open convex cone, $R > 0$, $d \geq 2$. Let $V = \Gamma \cap B(0, R)$. Let $E \subset \mathbb{R}^d$ be a nonempty neighborhood of 0. Define $W^+ \subset \mathbb{C}^d$, $W^- \subset \mathbb{C}^d$ by

$$(2.6) \quad W^+ = E + iV, \quad W^- = E - iV$$

Then there exists a fixed neighborhood U of $0 \in \mathbb{C}^d$ such that the following property holds: For any continuous function $g : W^+ \cup W^- \cup E \rightarrow \mathbb{C}$ that is holomorphic on $W^+ \cup W^-$, there is a holomorphic G on U such that $G|_{U \cap (W^+ \cup W^- \cup E)} = g$.

Proof: After composition with a linear isomorphism A in \mathbb{C}^d with real coefficients, we may assume that $\{(it_1, \dots, it_d) \in \mathbb{C}^d : t_j > 0, j = 1, \dots, d\} \subset A^{-1}(i\Gamma)$, there exists $B(0, R') \subset \mathbb{R}^d$ with $R' > 6\sqrt{d}$ and $E' = \{s \in \mathbb{R}^d : |s_j| < 6, j = 1, \dots, d\}$ such that $E' + iB(0, R') \subset A^{-1}(E + iB(0, R))$. So we reduce the case to the following:

$$\begin{aligned} E &= \{s \in \mathbb{R}^d : |s_j| < 6, j = 1, \dots, d\}, \\ V &= \{t \in \mathbb{R}^d : 0 < t_j < 6, j = 1, \dots, d\}, \\ W^+ &= E + iV, \quad W^- = E - iV. \end{aligned}$$

Let $U = D^d(0, 1)$. If $g : W^+ \cup W^- \cup E \rightarrow \mathbb{C}$ is continuous and g is holomorphic on $W^+ \cup W^-$, then there is a holomorphic G on U such that $G|_{U \cap (W^+ \cup W^- \cup E)} = g$. Let $c = \sqrt{2} - 1$ and define

$$(2.7) \quad \varphi : \overline{D}^2(0, 1) \rightarrow \mathbb{C} \quad (w, \lambda) \mapsto \frac{w + \lambda/c}{1 + c\lambda w}$$

Then

$$(2.8) \quad \operatorname{Im} \varphi(w, \lambda) = \frac{(1 - |\lambda|^2) \operatorname{Im}(cw) + (1 - |cw|^2) \operatorname{Im}(\lambda)}{c|1 + c\lambda w|^2}$$

Notice that

- (a) $\operatorname{sgn}(Im\varphi) = \operatorname{sgn}(Im\lambda)$ if $|\lambda| = 1$ or $w \in \mathbb{R}$;
- (b) $\varphi(w, 0) = w$;
- (c) $|\varphi(w, \lambda)| \leq (1 + 1/c)(1 - c) < 6$;
- (d) By (a), the function
 $\Phi : D^d(0, 1) \times \overline{D} \rightarrow \mathbb{C}^d$ with $\Phi(w, \lambda) = (\varphi(w_1, \lambda), \dots, \varphi(w_d, \lambda))$
satisfies $\Phi(w, e^{i\theta}) \in \operatorname{Dom}(g)$ for all $0 \leq \theta < 2\pi$, $w \in D^d(0, 1)$.

So by (d), we may define

$$(2.9) \quad G(w) = \frac{1}{2\pi} \int_0^{2\pi} g(\Phi(w, e^{i\theta})) d\theta, \quad w \in D^d(0, 1).$$

We claim that this is what we seek. First, G is holomorphic by an application of Morera's theorem. Next, for fixed $s \in E \cap D^d(0, 1)$, the function $g(\Phi(s, \cdot))$ is continuous on \overline{D} and holomorphic on $D - \mathbb{R}$ by (a), (c). Again, by Morera, $g(\Phi(s, \cdot))$ is holomorphic on all of D . It follows by Mean Value Property and (b) that

$$(2.10) \quad G(s) = \frac{1}{2\pi} \int_0^{2\pi} g(\Phi(s, e^{i\theta})) d\theta = g(\Phi(s, 0)) = g(s);$$

hence $G = g$ on $E \cap D^d(0, 1)$. If $s + it \in \mathbb{C}^d$ is fixed, $|s + it| < 1/2$, $t > 0$, then the function

$$(2.11) \quad \xi \mapsto G(s + \xi t) - g(s + \xi t)$$

is holomorphic for $\xi \in \mathbb{C}$ small and $\equiv 0$ when ξ is real. It follows that $G \equiv g$ on $W^+ \cup W^- \cup E$.

If we return to our original case, the desired extension of g is

$$(2.12) \quad G(w) = \frac{1}{2\pi} \int_0^{2\pi} g \circ A(\Phi(A^{-1}(w), e^{i\theta})) d\theta, \quad w \in A^{-1}(D^d(0, 1)).$$

since it agrees with our g in some open sets. U is taken to be $A^{-1}(D^d(0, 1))$ ■

Theorem 2.6. (Main theorem) Let $M \subset \mathbb{C}^{n+d}$ be a real analytic, generic submanifold. $0 \in M$. In a neighborhood Ω of 0 in \mathbb{C}^{n+d} , M is given by regular coordinates, $\rho := Im(w) - \phi(z, \bar{z}, Re(w)) = 0$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$. Let $f = u + iv$ be a holomorphic function defined in the wedge $W = W(\Omega, \rho, \Gamma)$, where $W(\Omega, \rho, \Gamma)$ is as above defined with Γ an open convex cone in \mathbb{R}^d . If f extends continuously up to the edge $M \cap \Omega$ with $v = Im(f)$ is real analytic on the edge $M \cap \Omega$. Then there exists a holomorphic function F in a neighborhood $\tilde{\Omega}$ of 0 in \mathbb{C}^{n+d} such that $F|_{W \cap \tilde{\Omega}} = f$.

Before proving the theorem, we need a lemma which ensures the existence of a wedge with edge as an open subset of \mathbb{R}^{2n+d} contained in $\tilde{\Psi}^{-1}(W)$. $\tilde{\Psi}$ is the real analytic diffeomorphism introduced in (2.3).

Lemma 2.7. There exists a neighborhood U_1 of 0 in \mathbb{R}^{2n+2d} , a neighborhood Q_1 of 0 in \mathbb{R}^{2n+d} , an open convex cone Γ' with vertex at 0 in \mathbb{R}^d such that $\tilde{\Psi}$ is a real analytic diffeomorphism from U_1 to $\tilde{\Psi}(U_1) \subset \Omega$ and $\tilde{\Psi}((Q_1 \times \Gamma') \cap U_1) \subset W$.

Proof of the lemma: Take a neighborhood U_1 of 0 in \mathbb{R}^{2n+2d} such that $\tilde{\Psi}$ is the real analytic diffeomorphism from U_1 to $\tilde{\Psi}(U_1) \subset \Omega$. $\tilde{\Psi}(z, s + it) = (z, s +$

$it + i\phi(z, \bar{z}, s + it)) = (\tilde{z}, \tilde{w})$, we can write $t = \Theta(\tilde{z}, \bar{\tilde{z}}, \tilde{w}, \bar{\tilde{w}})$ where Θ is \mathbb{R}^d valued, real analytic in all its variables. Note that $t := \Theta(\tilde{z}, \bar{\tilde{z}}, \tilde{w}, \bar{\tilde{w}})$ can be taken as a set of defining functions for M in $\tilde{\Psi}(U_1)$ because $\tilde{\Psi}$ is a real analytic diffeomorphism. By Proposition 2.4, there exists a neighborhood of 0 in \mathbb{C}^{n+d} (we may take it as a subset of $\Psi(U_1)$ and still denote it as $\Psi(U_1)$), a convex open cone Γ' such that

$$(2.13) \quad W(\Psi(U_1), t, \Gamma') \subset W = W(\Omega, \rho, \Gamma),$$

Take $Q_1 = U_1 \cap \mathbb{R}^{2n+d}$. Let $(\tilde{z}, \tilde{w}) = \tilde{\Psi}(x, t)$, where

$$(2.14) \quad (x, t) = (x_1, \dots, x_{2n+d}, t_1, \dots, t_d) \in (Q_1 \times \Gamma') \cap U_1.$$

Since $(t_1, \dots, t_d) \in \Gamma'$, thus $\Theta(\tilde{z}, \bar{\tilde{z}}, \tilde{w}, \bar{\tilde{w}}) \in \Gamma'$, $(\tilde{z}, \tilde{w}) \in W(\Psi(U_1), t, \Gamma') \subset W(\Omega, \rho, \Gamma)$. So we have $\tilde{\Psi}((Q_1 \times \Gamma') \cap U_1) \subset W$. ■

Proof of the theorem: $f|_{M \cap \Omega}$ is a continuous CR function on $M \cap \Omega$. By the lemma, the function $f \circ \tilde{\Psi}(z, \bar{z}, s + it) = f(z, \bar{z}, s + it + i\phi(z, \bar{z}, s + it))$ is defined on $\{(z, s + it) \in (Q_1 \times \Gamma') \cap U_1\}$, note that it is holomorphic in the variables $w = s + it$. To use Proposition 2.2, it suffices to show that $f \circ \tilde{\Psi}$ can be continuously extended to a full neighborhood of 0 in \mathbb{R}^{2n+2d} such that the extension is holomorphic in variables $w = s + it$.

Now, denote the restriction of the function $v \circ \tilde{\Psi}(z, s + it)$ on Q_1 to be $v(z, \bar{z}, s)$. Note that the real analytic diffeomorphism (holomorphic in $s + it$) $\tilde{\Psi}$ can be viewed as the flattening of the totally real submanifolds $\tilde{\Psi}(z, \bar{z}, s)$ for fixed z . By the real analyticity of v and Ψ , we can treat $v(z, \bar{z}, s)$ as a power series expansion at $0 \in \mathbb{R}^{2n+d}$ and complexify the variable s to w . Denote the new function to be $v(z, \bar{z}, w)$ defined in a product neighborhood $U_2 = U_3 \times U_4 \subset \mathbb{C}^n \times \mathbb{C}^d$ of 0. Note that $v(z, \bar{z}, w)$ is continuous in U_2 and holomorphic in variables w .

Following the idea of Theorem 1.2, the function $g(z, \bar{z}, w)$

$$(2.15) \quad := \begin{cases} \frac{f \circ \tilde{\Psi}(z, \bar{z}, w) - iv(z, \bar{z}, w)}{f \circ \tilde{\Psi}(z, \bar{z}, \bar{w}) - iv(z, \bar{z}, \bar{w})} & \text{for } (z, w) \in (Q_1 \times (\Gamma' \cup \{0\})) \cap U_2 \\ f \circ \tilde{\Psi}(z, \bar{z}, \bar{w}) - iv(z, \bar{z}, \bar{w}) & \text{for } (z, w) \in (Q_1 \times -\Gamma') \cap U_2 \end{cases}$$

is holomorphic in variables $w = s + it$ for fixed z and continuously up to $Q_1 \cap U_2$.

We shall apply the construction of holomorphic extension (2.12) in the proof of Theorem 2.5 to $g(z, \bar{z}, s + it)$ for every fixed z to obtain a continuous extension in all variables. According to the proof, we have a linear isomorphism A in \mathbb{C}^d with real coefficients and $\Phi : D^d(0, 1) \times \overline{D} \rightarrow \mathbb{C}^d$. Now we extend A to a linear isomorphism in \mathbb{C}^{n+d} which maps $(z, w) \in \mathbb{C}^{n+d}$ to $(z, A(w))$, still denote it as A . We also extend Φ to a mapping which takes $(z, w, \lambda) \in U_3 \times D^d(0, 1) \times \overline{D}$ to $(z, \Phi(w, \lambda)) \in \mathbb{C}^n \times \mathbb{C}^d$, still denote it as Φ . Since the choice of A depends only on U_4 and the cone Γ' , so we have the extension

$$(2.16) \quad G(z, \bar{z}, w) = \frac{1}{2\pi} \int_0^{2\pi} g \circ A(\Phi(A^{-1}(z, w), e^{i\theta})) d\theta, \quad \forall (z, w) \in A^{-1}(U_3 \times D^d(0, 1))$$

by Theorem 2.5 and Lemma 2.7.

From the construction (2.12), the extension $G(z, \bar{z}, s + it)$ is continuous on $A^{-1}(U_3 \times D^d(0, 1)) = U_3 \times A^{-1}(D^d(0, 1)) = U_3 \times U_5$ and holomorphic in variables w . Thus, the function $F(z, \bar{z}, s + it) := G + iv(z, \bar{z}, s + it)$ is a continuous extension of $f \circ \tilde{\Psi}(z, \bar{z}, s + it)$ in $U_3 \times U_5$, holomorphic in $w = s + it$. Thus by Proposition 2.2, f can be holomorphically extended to a full neighborhood $\tilde{\Omega}$ of 0 in \mathbb{C}^{n+d} . \blacksquare

The following theorem states that minimality is a sufficient condition for holomorphic extension of all CR functions from a generic submanifold M in \mathbb{C}^N into an open wedge. See [1], [4].

Theorem 2.8. Let M be a generic submanifold of \mathbb{C}^{n+d} of codimension d and $p_0 \in M$. If M is minimal at p_0 , then for every open neighborhood U of p_0 in M there exists a wedge W with edge M centered at p_0 such that every continuous CR function in U extends holomorphically to the wedge W .

Since a real analytic CR submanifold M in \mathbb{C}^N is finite type at $p_0 \in M$ if and only if it is minimal at p_0 . Thus, together with Theorem 2.6, we have Corollary 2.9.

Corollary 2.9. Let $M \subset \mathbb{C}^N$ be a real analytic, generic CR submanifold, finite type at $p_0 \in M$. If $f = u + iv$ is a continuous CR function defined in a neighborhood of p_0 in M with v real analytic. Then f can be holomorphically extended to a full neighborhood of p_0 in \mathbb{C}^N . (Or u is also real analytic in a neighborhood of p_0 in M by Corollary 2.3)

Corollary 2.10. Let M be a connected, real analytic, generic CR submanifold in \mathbb{C}^N . Assume M of finite type at its every point. If f, g are continuous CR functions on M such that $\text{Im}(f) = \text{Im}(g)$ on M and $f(p_0) = g(p_0)$ at $p_0 \in M$. Then $f = g$ on M .

Proof: $f - g$ is a continuous CR function on M . $\text{Im}(f - g) \equiv 0$ is real analytic. By finite type condition at every point and Corollary 2.9., $f - g$ can be extended to a holomorphic function H defined in a connected neighborhood U of M in \mathbb{C}^N .

We shall show that H is constant on U . Suppose not, without loss of generality, we assume that $\frac{\partial H}{\partial z_N}$ is not constant zero on U . Define subsets in U as follows:

$$(2.17) \quad Q := \{Z \in U : \frac{\partial H}{\partial z_N}(Z) = 0\}$$

$$(2.18) \quad I := \{Z \in U : \text{Im}(H)(Z) = 0\}$$

Clearly, $M \subset I$. Since M is a generic submanifold and Q is an analytic variety in U , $M \not\subset Q$. Let $p_1 \in M - Q$, then $\frac{\partial H}{\partial z_N}(p_1) \neq 0$. Thus we take a small neighborhood U_1 of p_1 in U such that $I \cap U_1$ is a smooth hypersurface and $U_1 \ni (z_1, \dots, z_{N-1}, z_N) \mapsto (z_1, \dots, z_{N-1}, H(Z))$ is a biholomorphism. This implies $I \cap U_1$ is biholomorphic equivalent to the real hyperplane which is of infinite type. However, $p_1 \in M \subset I$, M is of finite type at p_1 , hence $I \cap U_1$ is of finite type at p_1 . This contradiction shows that H is constant on U . By assumption, $H(p_0) = 0$ and

U is connected. Hence $H \equiv 0$ on U . So $f \equiv g$ on M . ■

Definition 2.11. A CR submanifold of the form

$$(2.19) \quad M = \{(z, s + it) \in \mathbb{C}^n \times \mathbb{C}^d; t = \phi(z, \bar{z})\}$$

where $\phi : \mathbb{C}^n \mapsto \mathbb{R}^d$ is smooth with $\phi(0) = 0$ and $d\phi(0) = 0$ is called rigid.

Corollary 2.12. Let M be a real analytic, generic submanifold, $0 \in M$, given by $\{(z, s + it) \in U \subset \mathbb{C}^n \times \mathbb{C}^d : t = \phi(z, \bar{z}, s)\}$. Let M' be a real analytic, generic, rigid submanifold, $0 \in M'$, given by $\{(z', s' + it') \in U' \subset \mathbb{C}^{n'} \times \mathbb{C}^{d'} : t' = \phi'(z', \bar{z}', s')\}$. If $H = (f_1, \dots, f_{n'}, g_1, \dots, g_{d'})$ is a holomorphic mapping defined in the wedge $W = W(U, \rho, \Gamma)$ with Γ an open convex cone. If H extends continuously to the edge $M \cap U$ with $H(0) = 0$, $H(M) \subset M'$ and $(f_1, \dots, f_{n'})$ can be holomorphically extended in a neighborhood of 0, then H can be holomorphically extended in a neighborhood of 0.

Proof: It suffices to show $Im(g_i)$ is real analytic for all i . By the assumption, $f_1, \dots, f_{n'}$ can be holomorphically extended in a neighborhood of 0. By Corollary 2.3, they are real analytic on M , hence their real and imaginary parts are real analytic on M . Since ϕ'_i is real analytic in a neighborhood of 0 in $\mathbb{C}^{n'}$ and $Im(g_i) = \phi'_i(Re(f), Im(f))$, hence $Im(g_i)$ are real analytic on M for all i . Thus by Theorem 2.6, g_i can be holomorphically extended to a neighborhood of 0, we get the desired result. ■

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